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Abstract: In this paper, we derive a one-dimensional asymptotic model for the dynamics of the magnetic moment in a twisted ferromagnetic nanowire with arbitrary elliptical cross-section, curvature and torsion.

Keywords: ferromagnetism, Landau-Lifschitz equation, nanowire, asymptotic process.

MSC: 35K55, 35Q60

1 Introduction

Ferromagnetic nanowires have promising potential applications in data storage devices (see [11]) and magnetic logic gates (see [1]). The wires used in these fields have complex shapes, with bends, notches or junctions. The study of domain wall formation and dynamics in such wires with complex geometry is crucial for applications (see [1, 11, 12, 13, 14, 16, 19]).
In this paper, we justify by asymptotic process a time-dependent one-dimensional model for a twisted, bent nanowire of variable cross-section. Our model takes into account the current effects.
Several works address the justification of 1d models for ferromagnetic nanowires (see [6], [15], [7]), however they consider a constant shaped cross-section or a cross-section oriented by the normal and binormal vectors of the curve modeling the wire central line. In this work we are able to consider here narrowing zones (arising in wires with notches) and twist shaped wires.

First, recall the 3d model for ferromagnetic materials (see [5, 10]). We consider a ferromagnetic body occupying the volume $\Omega \subset \mathbb{R}^3$. We denote by $\mathbf{M}(\mathbf{t},\mathbf{x})$ the magnetization distribution at the time $\mathbf{t}$ and at the point $\mathbf{x} \in \Omega$. At low temperature, the material satisfies the saturation constraint:

$$|\mathbf{M}(\mathbf{t},\mathbf{x})| = M_s, \tag{1.1}$$

where M_s is independent of $\mathbf{t}$ and $\mathbf{x}$. The variations of $\mathbf{M}$ satisfy the following Landau-Lifschitz-Gilbert equation:

$$\frac{\partial \mathbf{M}}{\partial \mathbf{t}} - \frac{\alpha}{M_s}\mathbf{M} \times \frac{\partial \mathbf{M}}{\partial \mathbf{t}} = -\gamma \mathbf{M} \times H_{\mathsf{eff}} - (\vec{\mathbf{u}} \cdot \nabla)\mathbf{M}, \tag{1.2}$$

where $\times$ is the cross product in $\mathbb{R}^3$, γ is the gyromagnetic constant, α is the damping coefficient. The effective field H_{eff} is given by:

$$H_{\mathsf{eff}} = \frac{A}{\mu_0 M_s^2}\Delta \mathbf{M} + \mathbf{H}_d(\mathbf{M}) + H_a, \tag{1.3}$$

where A is the exchange constant, μ_0 is the permeability of the vacuum, H_a is the applied magnetic field and $\mathbf{H}_d(\mathbf{M})$ is the demagnetizing field that deduced from $\mathbf{M}$ by the operator $\mathbf{H}_d$ given by:

$$\operatorname{curl} \mathbf{H}_d(\mathbf{M}) = 0 \quad \text{and} \quad \operatorname{div}(\mathbf{H}_d(\mathbf{M}) + \bar{\mathbf{M}}) = 0, \tag{1.4}$$

where $\bar{\mathbf{M}}(\mathbf{t},\mathbf{x}) = \mathbf{M}(\mathbf{t},\mathbf{x})$ for $\mathbf{x} \in \Omega$ and zero outside Ω.

The right-hand-side transport term in (1.2) describes the current effects: the velocity $\vec{\mathbf{u}}$ is given by

$$\vec{\mathbf{u}} = \frac{p\, g\, \mu_B}{2eM_s}\vec{J}, \tag{1.5}$$

where $\vec{J}$ is the current density, p, g, μ_B and e are respectively the current polarization, the Landé factor, the Bohr Magneton and the electron charge (see [4] and [17]).

Writing $\mathbf{M}(\mathbf{t},\mathbf{x}) = M_s\mathbf{m}(\gamma M_s\mathbf{t}\,,\mathbf{x})$, $H_a(\mathbf{t},\mathbf{x}) = M_s\mathbf{h}_a(\gamma M_s\mathbf{t},\mathbf{x})$, and $\vec{\mathbf{u}}(\mathbf{t},\mathbf{x}) = \gamma M_s\vec{\mathbf{v}}(\gamma M_s\mathbf{t},\mathbf{x})$, we obtain the rescaled model:

$$\begin{cases} \partial_t\mathbf{m} - \alpha\mathbf{m}\times\partial_t\mathbf{m} = -\mathbf{m}\times\mathbf{h}_{\text{eff}} - (\vec{\mathbf{v}}\cdot\nabla)\mathbf{m}, \\ \\ \mathbf{h}_{\text{eff}} = \ell^2\Delta\mathbf{m} + \mathbf{H}_d(\mathbf{m}) + \mathbf{h}_a. \end{cases} \tag{1.6}$$

where the dimensionless time is $t = \gamma M_s\mathbf{t}$ and $\ell^2 = \dfrac{A}{\mu_0 M_s^2}$.

We consider a curved ferromagnetic wire with non constant elliptical cross-section. We introduce $\mathbf{s}\mapsto\Gamma(\mathbf{s})$, the arc-length parametrization of the wire central line. We assume that for all $\mathbf{s}$, the cross-section is an ellipse whose axis are directed by $\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s})$ and $\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s})$, where for all $\mathbf{s}$, $(\Gamma'(\mathbf{s}),\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}),\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s}))$ forms a direct orthonormal basis of $\mathbb{R}^3$. We denote by $\eta\mathbf{a}(\mathbf{s})$ and $\eta\mathbf{b}(\mathbf{s})$ the associated semi-axis, where η is a small dimensionless parameter. Therefore, the wire is parametrized by:

$$\mathbf{x} = \Psi_\eta(\mathbf{s},u,v) := \Gamma(\mathbf{s}) + \eta\left(u\,\mathbf{a}(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s})\ + v\mathbf{b}(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s})\right), \tag{1.7}$$

where $(\mathbf{s},u,v)\in\mathcal{O} := [0,L]\times B_2(0,1)$. We denote by $B_2(0,1)$ the unit ball of $\mathbb{R}^2$ centered at 0 and radius 1 and by $\Omega_\eta = \Psi_\eta(\mathcal{O})$ the domain occupied by the ferromagnetic wire presented in figure 1.

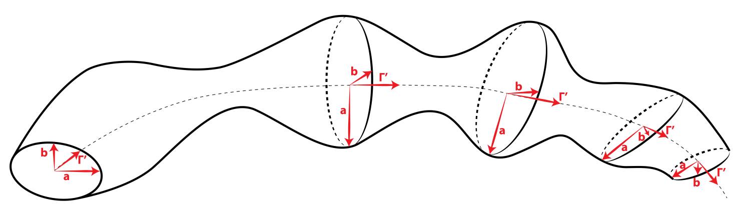

Figure 1: Ferromagnetic domain

We assume that $\Gamma\in\mathcal{C}^2([0,L])$, $\vec{\mathbf{e_a}}$ and $\vec{\mathbf{e_b}}$ are in $\mathcal{C}^1([0,L])$. We assume also that $\mathbf{a}$ and $\mathbf{b}$ are $\mathcal{C}^1$ on $[0,L]$ and bounded by below by a non negative constant.

Several Works study physically the effect of the geometrical form of the ferromagnetic wires on the magnetic moment behavior [18, 13, 8]. They study the effect of several geometrical aspects on the domain wall propagation. Respectively, they study the effect of the curvature, the torsion and the turning of the ribbon around its central wire on the domain wall propagation. All of these geometrical aspects are a particular cases of our geometry.

The existence of a global in time weak solution for (1.2) in Ω_η is established in [2] and [3]:

Proposition 1.1. *We fix $\eta > 0$. Let $\mathbf{m}_0^\eta \in H^1(\Omega_\eta;\mathbb{R}^3)$ satisfying $|\mathbf{m}_0^\eta| = 1$ a.e. Let $\mathbf{h}_{a,\eta}\in\mathcal{C}_b^0(\mathbb{R}^+;L^2(\Omega_\eta))$ and $\vec{\mathbf{v}}_\eta\in\mathcal{C}_b^0(\mathbb{R}^+;L^\infty(\Omega_\eta))$. There exists $\mathbf{m}^\eta:\mathbb{R}^+\times\Omega_\eta\longrightarrow\mathbb{R}^3$ satisfying the saturation constraint $|\mathbf{m}^\eta(t,\mathbf{x})| = 1$ a.e. such that*

– *for all $T\geq 0$, $\mathbf{m}^\eta\in L^\infty(0,T;H^1(\Omega_\eta))$ and $\dfrac{\partial\mathbf{m}^\eta}{\partial t}\in L^2([0,T]\times\Omega_\eta)$,*

– $\mathbf{m}^\eta(0,\cdot) = \mathbf{m}_0^\eta(\cdot)$ *in the trace sense,*

– *For all* $\Phi \in \mathbf{C}_c^\infty(\mathbb{R}^+; H^1(\Omega_\eta;\mathbb{R}^3))$,

$$\int_{\mathbb{R}^+\times\Omega_\eta} \left(\frac{\partial \mathbf{m}^\eta}{\partial t} - \alpha \mathbf{m}^\eta \times \frac{\partial \mathbf{m}^\eta}{\partial t}\right)\cdot \Phi dt\, d\mathbf{x} = \int_{\mathbb{R}^+\times\Omega_\eta} \sum_{i=1}^3 \ell^2 \left(\mathbf{m}^\eta \times \frac{\partial \mathbf{m}^\eta}{\partial \mathbf{x}_i}\right) \frac{\partial \Phi}{\partial \mathbf{x}_i}\, dt\, d\mathbf{x}$$
$$- \int_{\mathbb{R}^+\times\Omega_\eta} (\mathbf{H}_d(\mathbf{m}^\eta) + \mathbf{h}_{a,\eta})\cdot\Phi\, dt\, d\mathbf{x} - \int_{\mathbb{R}^+\times\Omega_\eta} (\vec{\mathbf{v}}_\eta\cdot\nabla)\mathbf{m}^\eta\cdot\Phi\, dt\, d\mathbf{x}, \tag{1.8}$$

– *for almost every* $t \geq 0$,

$$\mathcal{E}_\eta(\mathbf{m}^\eta)(t) + \frac{\alpha}{2}\int_0^t\int_{\Omega_\eta} |\frac{\partial \mathbf{m}^\eta}{\partial t}|^2 dt\, d\mathbf{x} \leq \mathcal{G}_\eta(t)\left(1 + \mathcal{V}_\eta(t)\exp\int_0^t \mathcal{V}_\eta(t)(s)ds\right) \tag{1.9}$$

where

$$\mathcal{E}_\eta(\mathbf{m}^\eta) = \frac{\ell^2}{2}\int_{\Omega_\eta} |\nabla \mathbf{m}^\eta|^2 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{H}_d(\mathbf{m}^\eta)|^2 d\mathbf{x},$$

with

$$\mathcal{G}_\eta(t) = \mathcal{E}(\mathbf{m}_0^\eta) + \int_0^t \|\mathbf{h}_{a,\eta}(\tau)\|^2_{L^2(\Omega_\eta)} d\tau \quad \textit{and} \quad \mathcal{V}_\eta(t)(t) = \frac{2}{\alpha\ell^2}\int_0^t \|\vec{\mathbf{v}}_\eta(\tau)\|^2_{L^\infty(\Omega_\eta)} d\tau.$$

We aim to obtain an asymptotic one-dimensional model for the wire when the parameter η tends to zero. By specifying the applied field and the electric current. We fix $h_a \in \mathcal{C}^0(\mathbb{R}^+; L^2([0,L];\mathbb{R}^3))$ and $j \in \mathcal{C}^0(\mathbb{R}^+; L^\infty([0,L]))$. We assume that the applied field is constant in the cross-section and that the current is constant in the cross-section and oriented in the direction of the wire, *i.e.*

$$\begin{cases} \mathbf{h}_{a,\eta}(t,\Psi_\eta(\mathbf{s},u,v)) = h_a(t,\mathbf{s}), \\ \vec{\mathbf{v}}_\eta(t,\Psi_\eta(\mathbf{s},u,v)) = j(t,\mathbf{s})\Gamma'(\mathbf{s}). \end{cases} \tag{1.10}$$

Then, we obtain the following theorem:

Theorem 1.1. *Let* $\mathbf{h}_{a,\eta} \in \mathcal{C}_b^0(\mathbb{R}^+; L^2(\Omega_\eta))$, $\vec{\mathbf{v}}_\eta \in \mathcal{C}_b^0(\mathbb{R}^+; L^\infty(\Omega_\eta))$ *defined by* (1.10). *Let* $m_0 \in H^1([0,L];S^2)$. *For* $\eta > 0$, *we define the initial data* $\mathbf{m}_0^\eta \in H^1(\Omega_\eta;S^2)$ *by:* $\mathbf{m}_0^\eta(\Psi_\eta(\mathbf{s},u,v)) = m_0(\mathbf{s})$. *We introduce the weak solution* $\mathbf{m}^\eta$ *of* (1.2) *given by Proposition 1.1 with initial data* $\mathbf{m}_0^\eta$. *We define* $m^\eta : \mathbb{R}^+ \times \mathcal{O} \longrightarrow S^2$ *by*

$$m^\eta(t,\mathbf{s},u,v) = \mathbf{m}^\eta(t,\Psi_\eta(\mathbf{s},u,v)).$$

Then when η *tends to zero, there exists a subsequence still denoted by* m^η *such that* $m^\eta \rightharpoonup m$ *in* $L^\infty(0,T;H^1(\mathcal{O}))$ *weak *. In addition,* m *does not depend on* u *and* v *and satisfies:*

– $|m(t,\mathbf{s})| = 1$ *a.e.,* $\partial_\mathbf{s} m \in L^\infty([0,T];L^2([0,L]))$ *and* $\partial_t m \in L^2([0,T]\times[0,L])$ *for all* T,

– $m(0,\mathbf{s}) = m_0(\mathbf{s})$ *in the trace sense,*

– *for all* $\phi \in \mathbf{C}_c^\infty(\mathbb{R}^+; H^1([0,L]))$,

$$\int_{\mathbb{R}^+\times[0,L]} \sigma\left(\frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t}\right)\cdot\phi\, dt\, d\mathbf{s} = \int_{\mathbb{R}^+\times[0,L]} \sigma\ell^2(m \times \frac{\partial m}{\partial \mathbf{s}})\cdot\frac{\partial\phi}{\partial\mathbf{s}}\, dt\, d\mathbf{s}$$
$$- \int_{\mathbb{R}^+\times[0,L]} \sigma\,(H_d(m) + h_a)\cdot\Psi\, dt\, d\mathbf{s} - \int_{\mathbb{R}^+\times[0,L]} \sigma j \partial_\mathbf{s} m\cdot\Phi\, dt\, d\mathbf{s},$$

where $\sigma(\mathbf{s}) = \pi\mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s})$ *and where the resulting demagnetizing field* $H_d(m)$ *is given by:*

$$H_d(m) = -\frac{\mathbf{b}}{\mathbf{a}+\mathbf{b}}(m\cdot\vec{\mathbf{e}}_\mathbf{a})\vec{\mathbf{e}}_\mathbf{a} - \frac{\mathbf{a}}{\mathbf{a}+\mathbf{b}}(m\cdot\vec{\mathbf{e}}_\mathbf{b})\vec{\mathbf{e}}_\mathbf{b}. \tag{1.11}$$

Remark 1.2. *From the physical point of view $\vec{\mathbf{u}}$ is proportional to the density current J (see (1.5)), it is natural to assume that the flux of $\vec{\mathbf{v}}_\eta$ through the cross-section is constant along the wire. For the asymptotic model, the corresponding assumption is that $\mathbf{s} \mapsto \sigma(\mathbf{s})j(\mathbf{s})$ is constant along the wire.*

Our one dimensional model is equivalent to smooth solutions of the model:

$$\frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t} = -m \times \left(\ell^2 \frac{\partial^2 m}{\partial \mathbf{s}^2} + \ell^2 \frac{\sigma'}{\sigma} \frac{\partial m}{\partial \mathbf{s}} + H_d(m) + h_a \right) - j \partial_{\mathbf{s}} m. \tag{1.12}$$

The localization of the demagnetizing field in (1.11) has been already observed in [6] and [15].
It can be more convenient to describe m in the mobile frame $(\Gamma'(\mathbf{s}), \vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}), \vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s}))$. For this purpose, we introduce r_1, r_2, r_3 in $\mathcal{C}^0([0, L])$ such that:

$$\begin{cases} \Gamma''(\mathbf{s}) = r_3(\mathbf{s}) \vec{\mathbf{e}_{\mathbf{a}}}(\mathbf{s}) - r_2(\mathbf{s}) \vec{\mathbf{e}_{\mathbf{b}}}(\mathbf{s}), \\ \\ \dfrac{d\vec{\mathbf{e}_{\mathbf{a}}}}{d\mathbf{s}}(\mathbf{s}) = -r_3(\mathbf{s}) \Gamma'(\mathbf{s}) + r_1(\mathbf{s}) \vec{\mathbf{e}_{\mathbf{b}}}(\mathbf{s}), \\ \\ \dfrac{d\vec{\mathbf{e}_{\mathbf{b}}}}{d\mathbf{s}}(\mathbf{s}) = r_2(\mathbf{s}) \Gamma'(\mathbf{s}) - r_1(\mathbf{s}) \vec{\mathbf{e}_{\mathbf{a}}}(\mathbf{s}). \end{cases} \tag{1.13}$$

We denote by $R(\mathbf{s}) = \begin{pmatrix} r_1(\mathbf{s}) \\ r_2(\mathbf{s}) \\ r_3(\mathbf{s})) \end{pmatrix}$ and by $\mathsf{m} = \begin{pmatrix} \mathsf{m}_1 \\ \mathsf{m}_2 \\ \mathsf{m}_3 \end{pmatrix}$ the coordinates of m in the mobile frame:

$$m(t, \mathbf{s}) = \mathsf{m}_1(t, \mathbf{s}) \Gamma'(\mathbf{s}) + \mathsf{m}_2(t, \mathbf{s}) \vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}) + \mathsf{m}_3(t, \mathbf{s}) \vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s}).$$

We have $|\mathsf{m}| = 1$ for all t and $\mathbf{s}$, and m satisfies (1.12) if and only if m satisfies:

$$\frac{\partial \mathsf{m}}{\partial t} - \alpha \mathsf{m} \times \frac{\partial \mathsf{m}}{\partial t} = -\mathsf{m} \times \mathcal{H}(\mathsf{m}) - j(\partial_{\mathbf{s}} \mathsf{m} + R \times \mathsf{m}),$$

with

$$\begin{aligned} \mathcal{H}(\mathsf{m}) = \quad & \ell^2 \frac{\partial^2 \mathsf{m}}{\partial \mathbf{s}^2} + 2\ell^2 R \times \frac{\partial \mathsf{m}}{\partial \mathbf{s}} + \ell^2 \frac{dR}{d\mathbf{s}} \times \mathsf{m} + \ell^2 R \times (R \times \mathsf{m}) + \ell^2 \frac{\sigma'}{\sigma} \left(\frac{\partial \mathsf{m}}{\partial \mathbf{s}} + R \times \mathsf{m} \right) \\ & - \frac{\mathbf{b}}{\mathbf{a} + \mathbf{b}} \mathsf{m}_2 - \frac{\mathbf{a}}{\mathbf{a} + \mathbf{b}} \mathsf{m}_3 + \mathsf{h}_a, \end{aligned}$$

where h_a are the coordinates of h_a in the mobile frame.

The paper is organized as follows. First, we achieve uniform bounds for m^η by writing the energy formula (1.9) using the new variables $(\mathbf{s}, u, v)$ in Section 2. Second, we take the limit of the formulation verified by m^η as η tends to zero in Section 3. To achieve this, we utilize the uniform bounds achieved in the preceding section and rewrite the weak formulation 1.8 in the new variables. Finally, the limit of the demagnetizing field is characterized in Section 4.

2 Uniform Estimates for the Rescaled Formulation

In this section, we aim to obtain uniform bounds for m^η, by rewriting the energy formula (1.9) in the variables $(t, \mathbf{s}, u, v)$.
We compute the differential of $\Psi_\eta : \mathcal{O} \longrightarrow \Omega_\eta$ given by (1.7) with respect to its variables and using (1.13), we obtain that:

$$\begin{aligned} \frac{\partial \Psi_\eta}{\partial \mathbf{s}} & = (1 - \eta (u\mathbf{a}(\mathbf{s}) r_1 - v\mathbf{b}(\mathbf{s}) r_2)) \Gamma'(\mathbf{s}) + \eta (u\mathbf{a}'(\mathbf{s}) - v\mathbf{b}(\mathbf{s}) r_3) \vec{\mathbf{e}}_{\mathbf{a}} + \eta (u\mathbf{a}(\mathbf{s}) r_3 + v\mathbf{b}'(\mathbf{s})) \vec{\mathbf{e}}_{\mathbf{b}} \\ & =: \Gamma'(s)(1 + \eta g_1) + \eta g_2 \vec{\mathbf{e}}_{\mathbf{a}} + \eta g_3 \vec{\mathbf{e}}_{\mathbf{b}}, \end{aligned} \tag{2.1}$$

where

$$g_1(\mathbf{s},u,v) = -u\mathbf{a}(\mathbf{s})r_1(\mathbf{s}) - v\mathbf{b}(\mathbf{s})r_2(\mathbf{s}),$$

$$g_2(\mathbf{s},u,v) = u\mathbf{a}'(\mathbf{s}) - v\mathbf{b}(\mathbf{s})r_3(\mathbf{s}),$$

$$g_3(\mathbf{s},u,v) = v\mathbf{b}'(\mathbf{s}) + u\mathbf{a}(\mathbf{s})r_3(\mathbf{s}).$$

Furthermore, we have

$$\frac{\partial \Psi}{\partial u} = \eta \mathbf{a}(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{a}} \quad \text{and} \quad \frac{\partial \Psi}{\partial v} = \eta \mathbf{b}(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{b}}.$$

Thus, we obtain:

$$\frac{\partial \Psi_\eta}{\partial \mathbf{s}} = \Gamma'(\mathbf{s})(1+\eta g_1) + \frac{g_2}{\mathbf{a}(\mathbf{s})}\frac{\partial \Psi_\eta}{\partial u} + \frac{g_3}{\mathbf{b}(\mathbf{s})}\frac{\partial \Psi_\eta}{\partial v}. \tag{2.2}$$

Then,

$$\Gamma'(\mathbf{s}) = \frac{1}{1+\eta g_1}\left(\frac{\partial \Psi_\eta}{\partial \mathbf{s}} - \frac{g_2}{\mathbf{a}}\frac{\partial \Psi_\eta}{\partial u} - \frac{g_3}{\mathbf{b}}\frac{\partial \Psi_\eta}{\partial v}\right), \quad \vec{\mathbf{e}}_{\mathbf{a}} = \frac{1}{\eta \mathbf{a}}\frac{\partial \Psi_\eta}{\partial u} \quad \text{and} \quad \vec{\mathbf{e}}_{\mathbf{b}} = \frac{1}{\eta \mathbf{b}}\frac{\partial \Psi_\eta}{\partial v}. \tag{2.3}$$

The Jacobian determinant of Ψ_η is given by:

$$J(s,u,v) = \eta^2(1+\eta g_1(s,u,v))\mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s}),$$

so that by changing the variable formula, for $f : \Omega_\eta \longrightarrow \mathbb{R}$ or $\mathbb{R}^3$, we have:

$$\int_{\Omega_\eta} f(\mathbf{x})\,d\mathbf{x} = \eta^2 \int_{\mathcal{O}} \mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s}) f(\Psi_\eta(\mathbf{s},u,v))\,(1+\eta g_1(\mathbf{s},u,v))\; d\mathbf{s}\,du\,dv. \tag{2.4}$$

In order to estimate $\|\nabla \mathbf{m}^\eta\|^2_{L^2(\Omega^\eta)}$, we remark that whatever the orthonormal frame $(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3)$,

$$|\nabla \mathbf{m}^\eta(\mathbf{x})|^2 = \sum_{i=1}^{3} |d\mathbf{m}^\eta(\mathbf{x})(\xi_i)|^2.$$

With the orthonormal frame $(\Gamma'(\mathbf{s}), \vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}), \vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s}))$, we obtain that

$$\begin{aligned} |\nabla \mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))|^2 = \quad & |d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\Gamma'(\mathbf{s}))|^2 + |d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}))|^2 \\ & + |d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s}))|^2. \end{aligned}$$

Since $m^\eta = \mathbf{m}^\eta \circ \Psi_\eta$, then using the chain rule, and (2.3), we obtain that:

$$d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\Gamma'(\mathbf{s})) = \frac{1}{1+\eta g_1}\left(\frac{\partial m^\eta}{\partial \mathbf{s}} - \frac{g_2}{\mathbf{a}}\frac{\partial m^\eta}{\partial u} - \frac{g_3}{\mathbf{b}}\frac{\partial m^\eta}{\partial v}\right),$$

$$d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e}_{\mathbf{a}}}) = \frac{1}{\eta \mathbf{a}}\frac{\partial m^\eta}{\partial u},$$

$$d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e}_{\mathbf{b}}}) = \frac{1}{\eta \mathbf{b}}\frac{\partial m^\eta}{\partial v}.$$

Thus, we have by (2.4) :

$$\begin{aligned} \int_{\Omega_\eta} |\nabla \mathbf{m}^\eta|^2 d\mathbf{x} = \quad & \eta^2 \int_{\mathcal{O}} \frac{\mathbf{ab}}{1+\eta g_1}\left|\frac{\partial m^\eta}{\partial \mathbf{s}} - \frac{g_2}{\mathbf{a}}\frac{\partial m^\eta}{\partial u} - \frac{g_3}{\mathbf{b}}\frac{\partial m^\eta}{\partial v}\right|^2 d\mathbf{s}\,du\,dv \\ & + \int_{\mathcal{O}} \mathbf{ab}(1+\eta g_1)\left(\frac{1}{\mathbf{a}^2}\left|\frac{\partial m^\eta}{\partial u}\right|^2 + \frac{1}{\mathbf{b}^2}\left|\frac{\partial m^\eta}{\partial v}\right|^2\right) d\mathbf{s}\,du\,dv. \end{aligned} \tag{2.5}$$

In particular, $\mathbf{m}_0^\eta$ does not depend on the transverse variables, so there exists a constant C_1 such that for all $\eta > 0$,

$$\mathcal{E}_\eta(m_0^\eta) \le C_1 \eta^2. \tag{2.6}$$

By properties of the applied field $\mathbf{h}_{a,\eta}$, we have for all t:

$$\|\mathbf{h}_{a,\eta}(t)\|^2_{L^2(\Omega_\eta)} = \eta^2 \int_{[0,L]} \frac{\pi \mathbf{ab}}{1+\eta g_1} |h_a(t,\mathbf{s})|^2 d\mathbf{s}.$$

So, using (2.6), there exists C_2 such that for all η,

$$\mathcal{G}_\eta(t) \le C_2 \eta^2 \left(1 + \int_0^t \|h_a(\tau,\cdot)\|^2_{L^2([0,L])} d\tau\right). \tag{2.7}$$

In addition, the current induced velocity $\vec{\mathbf{v}}_\eta(t)$ is uniformly bounded by $\|j(t,\cdot)\|_{L^\infty([0,L])}$. So there exists a constant C_3 such that for all t,

$$\mathcal{V}_\eta(t) \le C_3 \int_0^t \|j(\tau,\cdot)\|^2_{L^\infty([0,L])} d\tau := \nu(t). \tag{2.8}$$

We define G as:

$$G(t) = \left(C_1 + C_2 \int_0^t \|h_a(\tau,\cdot)\|_{L^2([0,L])} d\tau\right) (1 + \nu(t) \exp \nu(t)).$$

Therefore, using (2.7) and (2.8), we obtain by the energy formula that for all $t > 0$ and for all $\eta > 0$,

$$\frac{\ell^2}{2} \|\nabla \mathbf{m}^\eta(t,\cdot)\|^2_{L^2(\Omega_\eta)} + \frac{\alpha}{2} \int_0^t \left\| \frac{\partial \mathbf{m}^\eta}{\partial t}(\tau,\cdot)\right\|^2_{L^2(\Omega_\eta)} d\tau \le \eta^2 G(t).$$

Also, with (2.5) we get that for all T there exists a constant $C(T)$ such that for all $\eta > 0$,

$$\begin{aligned} \left\|\frac{\partial m^\eta}{\partial u}\right\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} + \left\|\frac{\partial m^\eta}{\partial v}\right\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} &\le C(T)\eta^2, \\ \left\|\frac{\partial m^\eta}{\partial \mathbf{s}} - g_2 \frac{\partial m^\eta}{\partial u} - g_3 \frac{\partial m^\eta}{\partial v}\right\|^2_{L^\infty(0,T;L^2(\mathcal{O}))} + \left\|\frac{\partial m^\eta}{\partial t}\right\|^2_{L^2([0,T]\times\mathcal{O})} &\le C(T). \end{aligned} \tag{2.9}$$

Concerning the demagnetizing field, for a given $w : \mathcal{O} \longrightarrow \mathbb{R}^3$, we define $\mathbf{w} : \Omega_\eta \longrightarrow \mathbb{R}^3$ by $\mathbf{w} \circ \Psi_\eta = w$, and we define $h_\eta(w) : \mathcal{O} \longrightarrow \mathbb{R}^3$ by:

$$\mathbf{H}_d(\mathbf{w}) \circ \Psi_\eta = h_\eta(w). \tag{2.10}$$

Since $-\mathbf{H}_d$ is an orthogonal projection for the $L^2(\mathbb{R}^3)$ inner product,then we have for all $w \in L^2(\mathcal{O})$:

$$\|\mathbf{H}_d(\mathbf{w})\|^2_{L^2(\Omega_\eta)} \le \|\mathbf{H}_d(\mathbf{w})\|^2_{L^2(\mathbb{R}^3)} \le \|\mathbf{w}\|^2_{L^2(\Omega_\eta)}, \tag{2.11}$$

so in the rescaled variables, we obtain that there exists a constant K such that for all w and all $\eta > 0$:

$$\|h_\eta(w)\|^2_{L^2(\mathcal{O})} \le K \|w\|^2_{L^2(\mathcal{O})}. \tag{2.12}$$

In addition, if $w \in L^\infty(\mathcal{O})$ takes its values in the unit sphere, then there exists a constant K' independent of w such that for all η,

$$\|h_\eta(w)\|^2_{L^2(\mathcal{O})} \le K'. \tag{2.13}$$

In particular, denoting by $H^\eta(t,\cdot) = h_\eta(m^\eta(t,\cdot))$, from the previous estimates, we obtain that for all $\eta > 0$,

$$\|H^\eta\|_{L^\infty(\mathbb{R}^+;L^2(\mathcal{O}))} \le K'. \tag{2.14}$$

Therefore, using Estimates (2.9) and (2.14), there exists a subsequence that we also denote by m^η such that for all T,

1. $m^\eta \rightharpoonup m$ in $L^\infty(0,T;H^1(\mathcal{O}))$ weak *,
2. $\dfrac{\partial m^\eta}{\partial u} \longrightarrow 0$ and $\dfrac{\partial m^\eta}{\partial u} \longrightarrow 0$ in $L^\infty(0,T;L^2(\mathcal{O})))$,
3. $\dfrac{\partial m^\eta}{\partial t} \rightharpoonup \dfrac{\partial m}{\partial t}$ weak in $L^2([0,T]\times\mathcal{O})$,
4. $H^\eta \rightharpoonup H$ in $L^\infty(0,T;L^2(\mathcal{O}))$ weak *.

By using the Aubin Simon Lemma, we conclude that m^η strongly tends to m in $L^\infty(0,T;L^p(\mathcal{O}))$ for $p\in[2,6[$ and by extracting a subsequence, we can assume that $m^\eta \longrightarrow m$ almost everywhere. In particular, we obtain that $|m|=1$ *a.e.* on $\mathbb{R}^+\times\mathcal{O}$.

3 Limit in the weak formulation

In this section, we aim to obtain the limit in the weak formulation by rewriting the formulation (1.8) in the new variables $(\mathbf{s},u,v)$ and taking the limit when η tends to zero. On the one hand, since the current $\vec{\mathbf{v}}_\eta$ is given by (1.10), we have

$$((\mathbf{v}_\eta\cdot\nabla)\mathbf{m}^\eta)(\Psi_\eta(\mathbf{s},u,v)) = j(t,\mathbf{s})d\mathbf{m}^\eta(\Psi_\eta(\mathbf{s},u,v))(\Gamma'(\mathbf{s})).$$

On the other hand, we remark that for all orthonormal frame $(\vec{\xi}_1,\vec{\xi}_2,\vec{\xi}_3)$, we have:

$$\sum_{i=1}^{3}\left(\mathbf{m}^\eta(\mathbf{x})\times\frac{\partial\mathbf{m}^\eta}{\partial x_i}(\mathbf{x})\right)\cdot\frac{\partial\Phi}{\partial x_i}(\mathbf{x}) = \sum_{i=1}^{3}\left(\mathbf{m}^\eta(\mathbf{x})\times d\mathbf{m}^\eta(\mathbf{x})(\vec{\xi}_i)\right)\cdot d\Phi(\mathbf{x})(\vec{\xi}_i).$$

We take $\vec{\xi}_1=\Gamma'(\mathbf{s})$, $\vec{\xi}_2=\vec{\mathbf{e}}_\mathbf{a}(\mathbf{s})$, $\vec{\xi}_3=\vec{\mathbf{e}}_\mathbf{b}(\mathbf{s})$, and Φ of the form $\Phi(t,\Psi_\eta(\mathbf{s},u,v))=\phi(t,\mathbf{s})$. Since ϕ depends only on $\mathbf{s}$, we have:

$$\frac{\partial\phi}{\partial u} = d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\frac{\partial\Psi_\eta}{\partial u}(\mathbf{s},u,v)) = \eta\mathbf{a}d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e_a}})=0,$$

$$\frac{\partial\phi}{\partial v} = d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\frac{\partial\Psi_\eta}{\partial v}(\mathbf{s},u,v)) = \eta\mathbf{b}d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e_b}})=0.$$

Thus, we obtain:

$$d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e_a}}) = d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\vec{\mathbf{e_b}})=0. \tag{3.1}$$

Next, differentiating ϕ with respect to $\mathbf{s}$ and by (2.1) and (3.1), we obtain:

$$\frac{\partial\phi}{\partial\mathbf{s}}(t,\mathbf{s}) = d\Phi(t,\Psi_\eta(\mathbf{s},u,v)(\frac{\Psi_\eta}{\partial\mathbf{s}}) = (1+\eta g_1)\,d\Phi(t,\Psi_\eta(\mathbf{s},u,v))(\Gamma'(\mathbf{s})).$$

Hence, we conclude that:

$$\begin{aligned}\sum_{i=1}^{3}\left(\mathbf{m}^\eta(\mathbf{x})\times d\mathbf{m}^\eta(\mathbf{x})(\vec{\xi}_i)\right)\cdot d\Phi(\mathbf{x})(\vec{\xi}_i) &= (\mathbf{m}^\eta(\mathbf{x})\times d\mathbf{m}^\eta(\mathbf{x})(\Gamma'(\mathbf{s})))\cdot d\Phi(\mathbf{x})(\Gamma'(\mathbf{s}))\\ &= \frac{1}{1+\eta g_1}\left(m^\eta(\mathbf{x})\times\left(\frac{\partial m^\eta}{\partial\mathbf{s}}-\frac{g_2}{\mathbf{a}}\frac{\partial m^\eta}{\partial u}-\frac{g_2}{\mathbf{b}}\frac{\partial m^\eta}{\partial u}\right)\right)\frac{\partial\phi}{\partial\mathbf{s}}.\end{aligned}$$

Then, we can write the weak formulation as follows:

$$\int_{\mathbb{R}^+\times\mathcal{O}} \mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s})(1+\eta g_1(\mathbf{s},u,v))\left(\frac{\partial m^\eta}{\partial t} - \alpha m^\eta \times \frac{\partial m^\eta}{\partial t}\right)\cdot \phi(t,\mathbf{s}) =$$

$$\int_{\mathbb{R}^+\times\mathcal{O}} \ell^2 \mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s})\left(m^\eta \times \left(\frac{\partial m^\eta}{\partial \mathbf{s}} - \frac{g_2}{\mathbf{a}}\frac{\partial m^\eta}{\partial u} - \frac{g_3}{\mathbf{b}}\frac{\partial m^\eta}{\partial v}\right)\right)\frac{\partial \phi}{\partial \mathbf{s}}$$

$$-\int_{\mathbb{R}^+\times\mathcal{O}} \mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s})(1+\eta g_1(\mathbf{s},u,v)) m^\eta \times (H^\eta + h_a)\cdot \phi(t,\mathbf{s})$$

$$-\int_{\mathbb{R}^+\times\mathcal{O}} \mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s}) j(t,\mathbf{s})\left(\frac{\partial m^\eta}{\partial \mathbf{s}} - \frac{g_2}{\mathbf{a}}\frac{\partial m^\eta}{\partial u} - \frac{g_3}{\mathbf{b}}\frac{\partial m^\eta}{\partial v}\right)\cdot \phi(t,\mathbf{s})$$

Taking the limit when η tends to zero and using that

- m^η tends to m strongly in $L^\infty(0,T;L^2(\mathcal{O}))$,
- $\dfrac{\partial m^\eta}{\partial \mathbf{s}}$ and H^η tend respectively to $\dfrac{\partial m}{\partial \mathbf{s}}$ and H in $L^\infty(0,T;L^2(\mathcal{O}))$ weak *,
- $\dfrac{\partial m^\eta}{\partial t}$ tends weakly to $\dfrac{\partial m}{\partial t}$ in $L^2(0,T;L^2(\mathcal{O}))$,
- $\dfrac{\partial m^\eta}{\partial u}$ and $\dfrac{\partial m^\eta}{\partial v}$ tend to zero strongly in $L^\infty(0,T;L^2(\mathcal{O}))$,
- m does not depend on the transverse variables u and v,

we obtain the following formulation:

$$\int_{\mathbb{R}^+\times[0,L]} \sigma\left(\frac{\partial m}{\partial t} - \alpha m\times\frac{\partial m}{\partial t}\right)\cdot\phi\, dt\, d\mathbf{s} = \int_{\mathbb{R}^+\times[0,L]} \ell^2\sigma\left(m\times\frac{\partial m}{\partial \mathbf{s}}\right)\cdot\frac{\partial\phi}{\partial \mathbf{s}}\, dt\, d\mathbf{s}$$

$$-\int_{\mathbb{R}^+\times[0,L]} \sigma m\times(H+h_a)\cdot\phi\, dt\, d\mathbf{s} - \int_{\mathbb{R}^+\times[0,L]} \sigma j\frac{\partial m}{\partial \mathbf{s}}\cdot\phi\, dt\, d\mathbf{s}.$$

where $\sigma(\mathbf{s}) = \pi\mathbf{a}(\mathbf{s})\mathbf{b}(\mathbf{s})$ is the rescaled area of the section. It remains to characterize the limit H of the rescaled demagnetizing field.

4 Limit for the Demagnetizing Field

Recall that we denoted by H^η the demagnetizing field induced by $\mathbf{m}^\eta$ written in the variables $(\mathbf{s},U)\in[0,L]\times B_2(0,1)$: $H^\eta(t,\mathbf{s},U) = \mathbf{H}_d(\mathbf{m}^\eta(t,\cdot))(\Psi^\eta(\mathbf{s},U))$. We introduce $H_\eta = h_\eta(m(t,\cdot))$. From (2.12), for all $\eta>0$:

$$\|H^\eta - H_\eta\|_{L^2(\mathcal{O})} \le K\|m^\eta - m\|_{L^2(\mathcal{O})}.$$

So in order to describe the limit of H^η, we will study the limit of H_η when η tends to zero. We claim the following Proposition.

Proposition 4.1. *Let $w\in\mathcal{C}^1(\overline{\mathcal{O}};\mathbb{R}^3)$. We assume that w is independent of the (u,v) variables, so that $w(\mathbf{s},u,v) = w_1(\mathbf{s})\Gamma'(\mathbf{s}) + w_2(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}) + w_3(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s})$. Then when η tends to zero, $h_\eta(w)$ tends strongly in $L^2(\mathcal{O})$ to $H_d(w)$ given by*

$$H_d(w) = -\frac{\mathbf{b}(\mathbf{s})}{\mathbf{a}(\mathbf{s})+\mathbf{b}(\mathbf{s})}(w\cdot\vec{\mathbf{e}}_{\mathbf{a}})\vec{\mathbf{e}}_{\mathbf{a}} - \frac{\mathbf{a}(\mathbf{s})}{\mathbf{a}(\mathbf{s})+\mathbf{b}(\mathbf{s})}(w\cdot\vec{\mathbf{e}}_{\mathbf{b}})\vec{\mathbf{e}}_{\mathbf{b}}.$$

Since m does not depend on the transverse variables, using the density of $\mathcal{C}^1([0,L])$ in $L^2([0,L])$, and (2.12), we obtain through Proposition 4.1 that H_η tends to $H_d(m)$ strongly in $L^2(\Omega)$, and we conclude the proof of Theorem 1.1. So it remains to establish Proposition 4.1.

Proof of Proposition 4.1: we define $\mathbf{w}_\eta : \Omega_\eta \longrightarrow \mathbb{R}^3$ by $\mathbf{w}_\eta \circ \Psi_\eta = w$. Recall that $h_\eta(w)$ is given by $h_\eta(w) = \mathbf{H}_d(\mathbf{w}_\eta) \circ \Psi_\eta$. Since $\operatorname{curl} \mathbf{H}_d(\mathbf{w}_\eta) = 0$, we can write $\mathbf{H}_d(\mathbf{w}_\eta) = -\nabla\phi$ with $\Delta\phi = \operatorname{div} \overline{\mathbf{w}_\eta}$, where $\overline{\mathbf{w}_\eta}$ is the extension of $\mathbf{w}_\eta$ by zero outside Ω_η. From the last equation, we have $\phi = -G{*}\operatorname{div} \overline{\mathbf{w}_\eta}$ where $G(x) = \dfrac{1}{4\pi|x|} \in L^1_{loc}(\mathbb{R}^3)$ is the fundamental solution for the operator $-\Delta$ in $\mathbb{R}^3$. Therefore we have:

$$\mathbf{H}_d(\mathbf{w}_\eta)(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega_\eta} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \operatorname{div} \mathbf{w}_\eta(\mathbf{y}) d\mathbf{y} + \frac{1}{4\pi}\int_{\partial\Omega_\eta} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \langle \mathbf{w}_\eta(\mathbf{y}) | \nu(\mathbf{y})\rangle d\sigma(\mathbf{y}), \tag{4.1}$$

where $\nu(\mathbf{y})$ is the outward unit normal on $\partial\Omega_\eta$.
Writting $w(\mathbf{s},u,v) = w_1(\mathbf{s})\Gamma'(\mathbf{s}) + w_2(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s}) + w_3(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s})$, then, by (2.3) we have:

$$\begin{aligned} w(\mathbf{s},u,v) =& w_1\left(\frac{1}{1+\eta g_1}\left(\frac{\partial\Psi_\eta}{\partial\mathbf{s}} - \frac{g_2}{\mathbf{a}}\frac{\partial\Psi_\eta}{\partial u} - \frac{g_3}{\mathbf{b}}\frac{\partial\Psi_\eta}{\partial v}\right)\right) + w_2\left(\frac{1}{\eta\mathbf{a}}\frac{\partial\Psi_\eta}{\partial u}\right) + w_3\left(\frac{1}{\eta\mathbf{b}}\frac{\partial\Psi_\eta}{\partial v}\right) \\ =&: \frac{\partial\Psi_\eta}{\partial\mathbf{s}}\gamma_1 + \frac{\partial\Psi_\eta}{\partial u}\gamma_2 + \frac{\partial\Psi_\eta}{\partial v}\gamma_3. \end{aligned}$$

Now, we compute Gram's matrix given by:

$$G(\mathbf{s},u,v) = \begin{pmatrix} \langle \frac{\partial\Psi_\eta}{\partial\mathbf{s}} | \frac{\partial\Psi_\eta}{\partial\mathbf{s}}\rangle & \langle \frac{\partial\Psi_\eta}{\partial\mathbf{s}} | \frac{\partial\Psi_\eta}{\partial u}\rangle & \langle \frac{\partial\Psi_\eta}{\partial\mathbf{s}} | \frac{\partial\Psi_\eta}{\partial v}\rangle \\ \langle \frac{\partial\Psi_\eta}{\partial\mathbf{s}} | \frac{\partial\Psi_\eta}{\partial u}\rangle & \langle \frac{\partial\Psi_\eta}{\partial u} | \frac{\partial\Psi_\eta}{\partial u}\rangle & \langle \frac{\partial\Psi_\eta}{\partial u} | \frac{\partial\Psi_\eta}{\partial v}\rangle \\ \langle \frac{\partial\Psi_\eta}{\partial\mathbf{s}} | \frac{\partial\Psi_\eta}{\partial v}\rangle & \langle \frac{\partial\Psi_\eta}{\partial u} | \frac{\partial\Psi_\eta}{\partial v}\rangle & \langle \frac{\partial\Psi_\eta}{\partial v} | \frac{\partial\Psi_\eta}{\partial v}\rangle \end{pmatrix} = \begin{pmatrix} (1+\eta g_1)^2 + \eta^2(g_2+g_3)^2 & \eta^2\mathbf{a}g_2 & \eta^2\mathbf{b}g_3 \\ \eta^2\mathbf{a}g_2 & \eta^2\mathbf{a}^2 & 0 \\ \eta^2\mathbf{b}g_3 & 0 & \eta^2\mathbf{b}^2 \end{pmatrix},$$

so that $\det \mathrm{G} = \eta^4\mathbf{a}^2\mathbf{b}^2(1+\eta g_1)^2$.

Since $w(\mathbf{s},u,v) = \dfrac{\partial\Psi_\eta}{\partial\mathbf{s}}\gamma_1 + \dfrac{\partial\Psi_\eta}{\partial u}\gamma_2 + \dfrac{\partial\Psi_\eta}{\partial v}\gamma_3$, then

$$\operatorname{div} \mathbf{w}_\eta(\Psi_\eta(s,u,v)) = \frac{1}{\sqrt{\det G}}\left(\frac{\partial}{\partial\mathbf{s}}(\gamma_1\sqrt{\det G}) + \frac{\partial}{\partial u}(\gamma_2\sqrt{\det G}) + \frac{\partial}{\partial v}(\gamma_3\sqrt{\det G})\right).$$

By direct substitution, we can see that

$$\begin{aligned} \operatorname{div} \mathbf{w}_\eta(\Psi_\eta(s,u,v)) =& \frac{1}{\eta^2\mathbf{ab}(1+g_1)}\left[\frac{\partial}{\partial\mathbf{s}}\left(\frac{w_1}{1+\eta g_1}\eta^2\mathbf{ab}(1+\eta g_1)\right)\right. \\ &+ \frac{\partial}{\partial u}\left(\left(\frac{w_2}{\eta\mathbf{a}} - \frac{g_2 w_1}{\mathbf{a}(1+g_1)}\right)\eta^2\mathbf{ab}(1+\eta g_1)\right) \\ &\left.+ \frac{\partial}{\partial v}\left(\left(\frac{w_2}{\eta\mathbf{a}} - \frac{g_2 w_1}{\mathbf{a}(1+g_1)}\right)\eta^2\mathbf{ab}(1+\eta g_1)\right)\right] \\ =& \frac{1}{\mathbf{ab}(1+\eta g_1)}\left(\frac{\partial(w_1\mathbf{ab})}{\partial\mathbf{s}} - w_1(\mathbf{ba}' + \mathbf{ab}') - r_1\mathbf{ab}w_2 - r_2\mathbf{ab}w_3\right). \end{aligned}$$

So, we obtain that:

$$\operatorname{div} \mathbf{w}_\eta(\Psi_\eta(\mathbf{s},U)) = \frac{1}{1+\eta g_1(\mathbf{s},U)}\left(\frac{\partial w_1}{\partial\mathbf{s}}(\mathbf{s}) - r_1(\mathbf{s})w_2(\mathbf{s}) - r_2(\mathbf{s})w_3(\mathbf{s})\right).$$

We split $\partial\Omega_\eta$ as $\Psi_\eta(\{0\}\times B_2(0,1)) \cup \Psi_\eta(\{L\}\times B_2(0,1)) \cup \Psi_\eta([0,L]\times\partial B_2(0,1)) := \Gamma_1\cup\Gamma_2\cup\Gamma_3$. So, we can rewrite (4.1) in the variables $(\mathbf{s},u,v)$ as follows:

$$H_d(\mathbf{w}_\eta)(\Psi(\mathbf{s},U)) = H_\eta(\mathbf{s},U) = I_1^\eta(\mathbf{s},U) + I_2^\eta(\mathbf{s},U) + I_3^\eta(\mathbf{s},U) + I_4^\eta(\mathbf{s},U),$$

where

- $I_1^\eta = -\dfrac{\eta^2}{4\pi}\displaystyle\int_{\mathcal{O}} \dfrac{\Psi_\eta(\mathbf{s},U)-\Psi_\eta(\mathbf{s}',U')}{|\Psi_\eta(\mathbf{s},U)-\Psi_\eta(\mathbf{s}',U')|^3}\left(\dfrac{\partial w_1}{\partial \mathbf{s}}(\mathbf{s}')-r_1(\mathbf{s}')w_2(\mathbf{s}')-r_2(\mathbf{s}')w_3(\mathbf{s}')\right)\sigma(\mathbf{s}')d\mathbf{s}'dU'$
- $I_2^\eta = -\dfrac{\eta^2}{4\pi}\displaystyle\int_{B_2(0,1)} \dfrac{\Psi_\eta(s,U)-\Psi_\eta(0,U')}{|\Psi_\eta(s,U)-\Psi_\eta(0,U')|^3}\sigma(0)w_1(0)dU',$
- $I_3^\eta = \dfrac{\eta^2}{4\pi}\displaystyle\int_{B_2(0,1)} \dfrac{\Psi_\eta(s,U)-\Psi_\eta(L,U')}{|\Psi_\eta(s,U)-\Psi_\eta(L,U')|^3}\sigma(L)w_1(L)dU',$
- $I_4^\eta = \dfrac{\eta}{4\pi}\displaystyle\int_0^L\int_0^{2\pi} \dfrac{\Psi_\eta(\mathbf{s},U)-\Psi_\eta(\mathbf{s}',\vec{\mathbf{e}}_r(\theta))}{|\Psi_\eta(\mathbf{s},U)-\Psi_\eta(\mathbf{s}',\vec{\mathbf{e}}_r(\theta))|^3}A(\mathbf{s}',\theta)G_\eta(\mathbf{s}',\theta)d\mathbf{s}'d\theta,$

with

$$\vec{\mathbf{e}}_r(\theta)=(\cos\theta,\sin\theta),\quad A(\mathbf{s}',\theta)=\frac{\mathbf{b}(\mathbf{s}')w_2(\mathbf{s}')\cos\theta+\mathbf{a}(\mathbf{s}')w_3(\mathbf{s}')\sin\theta}{(\mathbf{b}^2(\mathbf{s}')\cos^2\theta+\mathbf{a}^2(\mathbf{s}')\sin^2\theta)^{\frac12}},$$

and $G_\eta(\mathbf{s}',\theta)=\left(\alpha_1(\mathbf{s}',\theta)\alpha_2(\mathbf{s}',\theta)-(\alpha_3(\mathbf{s}',\theta))^2\right)^{\frac12}$.

Lemma 4.1. *There exists $C>0$ and $\eta_0>0$ such that for all $\eta\le\eta_0$, $\mathbf{s},\mathbf{s}'\in[0,L]$ and $U,U'\in B_2(0,1)$ we have*

$$\|\Psi_\eta(\mathbf{s},U)-\Psi_\eta(\mathbf{s}',U')\|_{\mathbb{R}^3}\ge C(|\mathbf{s}-\mathbf{s}'|^2+\eta^2\|U-U'\|^2)^{\frac12}.$$

Proof of Lemma 4.1. Assume that this Lemma is false. Then for all $n\ge 0$, there exists $\eta_n\le\frac1n$, $\mathbf{s}_n$, $\mathbf{s}'_n\in[0,L]$ and U_n, $U'_n\in B_2(0,1)$ such that

$$\|\Psi_{\eta_n}(\mathbf{s}_n,U_n)-\Psi_h(\mathbf{s}'_n,U'_n)\|<\frac1n(|\mathbf{s}_n-\mathbf{s}'_n|^2+(\eta_n)^2\|U_n-U'_n\|^2)^{\frac12},\tag{4.2}$$

which implies that $\|X(\mathbf{s}_n)-X(\mathbf{s}'_n)\|$ tends to zero. By extracting a subsequence we can assume that $\mathbf{s}_n$ and $\mathbf{s}'_n$ tend respectively to $\mathbf{s}_\infty$ and $\mathbf{s}'_\infty$. Since $\|X(\mathbf{s}_n)-X(\mathbf{s}'_n)\|\longrightarrow 0$, then $X(\mathbf{s}_\infty)=X(\mathbf{s}'_\infty)$, which implies that $s_\infty=s'_\infty$. Therefore $(\mathbf{s}_n,\eta_nU_n)_n$, $(\mathbf{s}'_n,\eta_nU'_n)_n$ tends to $(\mathbf{s}_\infty,0)$.
Next we define $\Psi:(\mathbf{s},U)\mapsto\Gamma(s)+u\mathbf{a}(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{a}}(\mathbf{s})+v\mathbf{b}(\mathbf{s})\vec{\mathbf{e}}_{\mathbf{b}}(\mathbf{s})$. So, by the local inversion theorem, there exists $\nu>0$ such that Ψ is a $\mathcal{C}^1$-diffeomorphism from $[\mathbf{s}_\infty-\nu,\mathbf{s}_\infty+\nu]\times B_2(0,\nu)$ into its range $\mathcal{V}$. Even if it means reducing ν, we can assume that Ψ^{-1} is Lipschitz on $\mathcal{V}$:

$$\exists C,\ \forall(\mathbf{x},\mathbf{y})\in\mathcal{V}\times\mathcal{V},\ \|\Psi^{-1}(\mathbf{x})-\Psi^{-1}(\mathbf{y})\|\le C\|\mathbf{x}-\mathbf{y}\|.\tag{4.3}$$

For n large enough, $(\mathbf{s}_n,\eta_nU_n)$ and $(\mathbf{s}'_n,\eta_nU'_n)$ are in $[\mathbf{s}_\infty-\nu,\mathbf{s}_\infty+\nu]\times B_2(0,\nu)$, so by applying (4.3) for $\mathbf{x}=\Psi_{\eta_n}(\mathbf{s}_n,U_n)$ and $\mathbf{y}=\Psi_{\eta_n}(\mathbf{s}'_n,U'_n)$, we obtain that

$$\left(|\mathbf{s}_n-\mathbf{s}'_n|^2+(\eta_n)^2\|U_n-U'_n\|^2\right)^{\frac12}\le C\|\Psi_{\eta_n}(\mathbf{s}_n,U_n)-\Psi_h(\mathbf{s}'_n,U'_n)\|$$

which together with (4.2) implies that $1<C\frac1n$ for n large enough, which is a contradiction. This concludes the proof of Lemma 4.1.

□

We first prove that I_2^η and I_3^η tend to zero in $L^2(\mathcal{O})$ when η tends to zero. By Lemma 4.1,

$$\begin{aligned}|I_2^\eta(\mathbf{s},U)|\le\ & K\eta^2\int_{B_2(0,1)}\frac{1}{\mathbf{s}^2+\eta^2\|U-U'\|^2}dU' &\le\ & K\eta^2\int_{B_2(0,2)}\frac{1}{\mathbf{s}^2+\eta^2\|U'\|^2}dU'\\ \le\ & 2\pi K\eta^2\int_0^2\frac{r}{\mathbf{s}^2+\eta^2r^2}dr &\le\ & \pi K\left(\ln(\mathbf{s}^2+4\eta^2)-\ln\mathbf{s}^2\right).\end{aligned}$$

Clearly, the right hand side of the previous inequality strongly tends to zero in $L^2(\mathcal{O})$ when η tends to zero, so I_2^η tends to zero in $L^2(\mathcal{O})$. In the same way, we prove the same result for I_3^η.

Since $w \in \mathcal{C}^1(\overline{\mathcal{O}})$, we can bound I_1^η by the same arguments:

$$|I_1^\eta(\mathbf{s},U)| \leq K\eta^2 \int_0^L \int_{B_2(0,1)} \frac{1}{(\mathbf{s}-\mathbf{s}')^2+\eta^2\|U-U'\|^2} dU' d\mathbf{s}' \leq 2\pi K \int_0^L \left(\ln((\mathbf{s}')^2+4\eta^2) - \ln(\mathbf{s}')^2\right) d\mathbf{s}'.$$

The right hand side of the previous estimate does not depend on $(\mathbf{s}, U)$ and tends to zero when η tends to zero, so I_1^η tends to zero uniformly on Ω, and it strongly tends to zero in $L^2(\Omega)$.

Now, we split I_4^η in 2 parts: $I_4^\eta = I_{4,1}^\eta + I_{4,2}^\eta$ with:

$$I_{4,1}^\eta = \frac{\eta}{4\pi} \int_0^{2\pi} \int_0^L \frac{\Psi_\eta(\mathbf{s}, \vec{\mathbf{e}}_r(\theta)) - \Psi_\eta(\mathbf{s}', \vec{\mathbf{e}}_r(\theta))}{|\Psi_\eta(\mathbf{s},U) - \Psi_\eta(\mathbf{s}', \vec{\mathbf{e}}_r(\theta))|^3} A(\mathbf{s}',\theta) G_\eta(\mathbf{s}',\theta) d\mathbf{s}' d\theta,$$

$$I_{4,2}^\eta = \frac{\eta^2}{4\pi} \int_0^{2\pi} \int_0^L \frac{\mathbf{a}(\mathbf{s})\vec{\mathbf{e}}_\mathbf{a}(\mathbf{s})(u-\cos\theta) + \mathbf{b}(\mathbf{s})\vec{\mathbf{e}}_\mathbf{b}(\mathbf{s})(v-\sin\theta)}{|\Psi_\eta(\mathbf{s},U) - \Psi_\eta(\mathbf{s}', \vec{\mathbf{e}}_r(\theta))|^3} A(\mathbf{s}',\theta) G_\eta(\mathbf{s}',\theta) d\mathbf{s}' d\theta.$$

By Lemma 4.1, we have:

$$\begin{aligned}
|I_{4,1}^\eta| &\leq K\eta \int_0^{2\pi} \int_0^L \frac{|\mathbf{s}-\mathbf{s}'| d\theta d\mathbf{s}'}{(\mathbf{s}-\mathbf{s}')^2 + \eta^2\|U-\vec{\mathbf{e}}_r(\theta)\|^2} \leq 2K\eta \int_0^{2\pi} \int_0^L \frac{\mathbf{s}'\, d\theta d\mathbf{s}'}{(\mathbf{s}')^2 + \eta^2\|U-\vec{\mathbf{e}}_r(\theta)\|^2} \\
&\leq 4\pi K\eta \int_0^L \frac{\mathbf{s}'}{(\mathbf{s}')^2 + \eta^2(\|U\|-1)^2} d\mathbf{s}' d\theta \quad \leq 2\pi\eta \left(\ln(L+\eta^2(\|U\|-1)^2) - \ln \eta^2(\|U\|-1)^2\right)\right) \\
&\leq 2\pi\eta \ln(L+2) - 4\pi\eta\ln\eta - 4\pi\eta\ln(1-\|U\|).
\end{aligned}$$

When η tends to zero, the right hand side term tends to zero in $L^2(\mathcal{O})$ so $I_{4,1}^\eta$ tends to zero strongly in $L^2(\mathcal{O})$.

For the last term, by Taylor expansion, we write $\Gamma(\mathbf{s}') - \Gamma(\mathbf{s}) = (\mathbf{s}'-\mathbf{s})\Lambda(\mathbf{s},\mathbf{s}')$ where $\Lambda(\mathbf{s},\mathbf{s}') = \int_0^1 \Gamma'(\mathbf{s}+\tau(\mathbf{s}'-\mathbf{s}))\, d\tau$ so that $\Lambda \in \mathcal{C}^1([0,L]^2;\mathbb{R}^3)$. In addition, we denote by $\chi(\mathbf{s})$ the 3×2 matrix such that $\chi(\mathbf{s})(u,v) = \mathbf{a}(\mathbf{s})\vec{\mathbf{e}}_\mathbf{a}(\mathbf{s})u + \mathbf{b}(\mathbf{s})\vec{\mathbf{e}}_\mathbf{b}(\mathbf{s})v$.
Using change of variable $\mathbf{s}' = \mathbf{s} + \tau\eta\|U - \vec{\mathbf{e}}_r(\theta)\|$ in the integral in $\mathbf{s}'$, we obtain:

$$I_{4,2}^\eta(\mathbf{s},U) = \frac{1}{4\pi} \int_0^{2\pi} \int_{\frac{-\mathbf{s}}{\eta\|U-\vec{\mathbf{e}}_r(\theta)\|}}^{\frac{L-s}{\eta||U-\vec{\mathbf{e}}_r(\theta)||}} K_\eta(\mathbf{s},U,\theta,\tau)\, d\tau\, d\theta,$$

where

$$\begin{aligned}
K_\eta(\mathbf{s},U,\theta,\tau) &= \frac{\eta^3|U-\vec{\mathbf{e}}_r(\theta)|\chi(\mathbf{s})(U-\vec{\mathbf{e}}_r(\theta))A(\mathbf{s}+\eta\tau|U-\vec{\mathbf{e}}_r(\theta)|,\theta)G_\eta(\mathbf{s}+\eta\tau|U-\vec{\mathbf{e}}_r(\theta)|,\theta)}{|\Psi_\eta(\mathbf{s}+\tau\eta|U-\vec{\mathbf{e}}_r(\theta)|, \vec{\mathbf{e}}_r(\theta)) - \Psi_\eta(\mathbf{s},U)|^3} \\
&= \frac{|U-\vec{\mathbf{e}}_r(\theta)|\chi(\mathbf{s})(U-\vec{\mathbf{e}}_r(\theta))A(\mathbf{s}+\eta\tau|U-\vec{\mathbf{e}}_r(\theta)|,\theta)G_\eta(\mathbf{s}+\eta\tau|U-\vec{\mathbf{e}}_r(\theta)|,\theta)}{\left|\tau|U-\vec{\mathbf{e}}_r(\theta)|\Lambda(\mathbf{s},\mathbf{s}+\tau\eta|U-\vec{\mathbf{e}}_r(\theta)|) + \chi(\mathbf{s}+\tau\eta|U-\vec{\mathbf{e}}_r(\theta)|)(\vec{\mathbf{e}}_r(\theta)) - \chi(\mathbf{s})(U)\right|^3}.
\end{aligned}$$

Hence, using Lemma 4.1, and since A and G_η are uniformly bounded, then there exists a constant M independent of η, $\mathbf{s}$, U, θ, and τ such that :

$$|K_\eta(\mathbf{s},U,\theta,\tau)| \leq M \frac{\eta^3|U-\vec{\mathbf{e}}_r(\theta)|^2}{(|\tau^2\eta^2|U-\vec{\mathbf{e}}_r(\theta)|^2 + \eta^2|U-\vec{\mathbf{e}}_r(\theta)|^2)^{\frac{3}{2}}} \leq \frac{M}{|U-\vec{\mathbf{e}}_r(\theta)|} \frac{1}{(1+\tau^2)^{\frac{3}{2}}}. \tag{4.4}$$

Therefore, when η tends to zero, for a fixed $(\mathbf{s},U,\theta,\tau)$, $K_\eta(\mathbf{s},U,\theta,\tau)$ tends to

$$K_0(\mathbf{s},U,\theta,\tau) := \frac{\chi(\mathbf{s})(U-\vec{\mathbf{e}}_r(\theta))A(\mathbf{s},\theta)}{(|U-\vec{\mathbf{e}}_r(\theta)|^2\tau^2 + |\chi(\mathbf{s})(U-\vec{\mathbf{e}}_r(\theta))|^2)^{\frac{3}{2}}} G_0(\mathbf{s},\theta),$$

since $\Lambda(\mathbf{s},\mathbf{s}) = \Gamma'(\mathbf{s})$ is orthogonal to the range of $\chi(\mathbf{s})$.

With Estimate (4.4), we obtain by the Lebesgue dominated convergence theorem that for all $(\mathbf{s}, U)$, $I^{\eta}_{4,2}(\mathbf{s}, U)$ tends to

$$D(\mathbf{s}, U) = \frac{1}{4\pi}\int_0^{2\pi}\int_{\mathbb{R}} K_0(\mathbf{s}, U, \theta, \tau)d\tau\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\chi(\mathbf{s})(U - \vec{\mathbf{e}}_r(\theta))A(\mathbf{s}, \theta)}{|\chi(\mathbf{s})(U - \vec{\mathbf{e}}_r(\theta))|^2}G_0(\mathbf{s}, \theta)d\theta.$$

In addition, using Estimate (4.4) we obtain that for all η, and for all $(\mathbf{s}, U) \in \mathcal{O}$,

$$|I^{\eta}_{4,2}(\mathbf{s}, U)| \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{M}{|U - \vec{\mathbf{e}}_r(\theta)|}d\theta. \tag{4.5}$$

Using that $U = re^{i\zeta}$, such that $(\zeta, r) \in [0, 2\pi] \times B(0, 1)$ and by changing the variable $v = \theta - \zeta$ we obtain

$$\int_0^{2\pi}\frac{M}{|U - \vec{\mathbf{e}}_r(\theta)|}d\theta = \int_0^{2\pi}\frac{M}{|r - e^{i(\theta-\zeta)}|}d\theta = \int_{-\zeta}^{2\pi-\zeta}\frac{M}{|r - e^{iv}|}dv.$$

Since the function $v \to r - e^{iv}$ is 2π-periodic, we get:

$$\int_0^{2\pi}\frac{M}{|U - \vec{\mathbf{e}}_r(\theta)|}d\theta = \int_0^{2\pi}\frac{M}{\sqrt{(r - \cos\theta)^2 + \sin^2\theta}}d\theta.$$

In order to show the right hand of (4.5) is in $L^2(\mathcal{O})$, we have the following Lemma.

Lemma 4.2. *There exists a constant $C > 0$, such that for all $(\theta, r) \in [0, 2\pi] \times B(0, 1)$, we have*

$$(r - \cos\theta)^2 + \sin^2\theta \geq C((r - 1)^2 + \theta^2).$$

Proof of Lemma 4.2 Denoting by $g(\theta, r) = (r - \cos\theta)^2 + \sin^2\theta$, note that

$$g(0, 1) = 0, \quad \nabla g(0, 1) = 0 \quad \text{and } \operatorname{Hess} g(0, 1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then by Taylor expansion in the neighbourhood of $(0, 1)$, so that for all $(\theta, r) \in B((0, 1), \nu)$

$$g(\theta, r) = \int_0^1 (\theta, r - 1)^T \operatorname{Hess} g((0, 1) + t(\theta, r - 1))(\theta, r - 1)(1 - t)dt.$$

Since the $\operatorname{Hess} g(0, 1)$ is strictly positive, there exists $C_1 > 0$, such that

$$g(\theta, r) \geq C_1(\theta^2 + (r - 1)^2).$$

We remark that the function $\dfrac{g(\theta, r)}{(\theta^2 + (r - 1)^2)}$ is continuous and positive, furthermore $[0, 2\pi]\times\overline{B(0, 1)} - B((0, 1), \nu)$ is compact, so there exists C_2 such that

$$\frac{g(\theta, r)}{(\theta^2 + (r - 1)^2)} \geq C_2$$

taking $C = \min\{C_1, C_2\}$, we conclude the proof of our Lemma. □

Now, using the above Lemma, we obtain:

$$\int_0^{2\pi}\frac{M}{|U - \vec{\mathbf{e}}_r(\theta)|}d\theta \leq \int_0^{2\pi}\frac{M}{\sqrt{(r - 1)^2 + \theta^2}}d\theta.$$

Using the change of variable $u = \dfrac{\theta}{|r - 1|}$, yields:

$$\int_0^{2\pi}\frac{1}{\sqrt{(r - 1)^2 + \theta^2}}d\theta \leq \int_0^{\frac{2\pi}{|a-1|}}\frac{1}{\sqrt{1 + u^2}}du \leq \int_0^1 du + \int_1^{\frac{2\pi}{|a-1|}}\frac{1}{u}du.$$

Then, we conclude that

$$\int_0^{2\pi} \frac{M}{|U - \vec{\mathbf{e}}_r(\theta)|} d\theta \le M(1 + \log(\frac{2\pi}{|r-1|})) \in L^2(\mathcal{O}).$$

Furthermore $|I^\eta_{4,2}|$ strongly tends to $D(\mathbf{s}, U)$ in $L^2(\mathcal{O})$.
We denote $X = u\mathbf{a}(\mathbf{s}) + iv\mathbf{b}(\mathbf{s})$. We remark that $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$, with $z = e^{i\theta}$. So $D(\mathbf{s}, U)$ can be written as an integral of a meromorphic function F on the circle $\mathcal{C}(0,1)$ of center 0 and radius 1:

$$D(\mathbf{s}, U) = \frac{1}{2i\pi} \int_{\mathcal{C}(0,1)} F(z) dz,$$

with

$$F(z) = \frac{(\mathbf{a}(\mathbf{s})w_2(\mathbf{s}) + i\mathbf{b}(\mathbf{s})w_3(\mathbf{s}))\, z^2 + \mathbf{a}(\mathbf{s})w_2(\mathbf{s}) + iw_3(\mathbf{s})\mathbf{b}(s)}{z\left((\mathbf{b}(\mathbf{s}) - \mathbf{a}(\mathbf{s}))z^2 + 2\bar{X}z - (\mathbf{a}(\mathbf{s}) + \mathbf{b}(\mathbf{s}))\right)}.$$

In [9], using complex analysis arguments, Jizzini proves the following proposition:

Proposition 4.2. *The meromorphic function F has only one pole $z = 0$ and its residue is given by:*

$$Res_F(0) = \frac{\mathbf{b}(\mathbf{s})w_2(\mathbf{s}) + iw_3(\mathbf{s})\mathbf{a}(\mathbf{s})}{-(\mathbf{a}(s) + \mathbf{b}(\mathbf{s}))}.$$

For the convenience of the reader, we reproduce here the proof of the proposition (see [9]).
Proof of Proposition 4.2. Let us suppose that $z \neq 0$ is a pole of F inside $C(0,1)$ which means z verifies the following equation

$$(\mathbf{b}(\mathbf{s}) - \mathbf{a}(\mathbf{s}))z^2 + 2\bar{X}z - (\mathbf{a}(\mathbf{s}) + \mathbf{b}(\mathbf{s})) = 0. \tag{4.6}$$

Denoting $\delta(\mathbf{s}) = \dfrac{\mathbf{b}(\mathbf{s})}{\mathbf{a}(\mathbf{s})}$, we can rewrite (4.6) as follows:

$$(\delta(\mathbf{s}) - 1)z + \frac{\bar{X}}{\mathbf{a}(\mathbf{s})} - (\delta(\mathbf{s}) + 1)\frac{1}{z} = 0. \tag{4.7}$$

Since X is inside the ellipse $E(\mathbf{a}(\mathbf{s}), \delta(\mathbf{s})\mathbf{a}(\mathbf{s}))$, there exists $\lambda_0(\mathbf{s}) \in [0,1[$ and $\theta_0(\mathbf{s}) \in]-\pi, \pi[$ such that

$$\frac{\bar{X}}{\mathbf{a}(\mathbf{s})} = \lambda_0 \cos\theta_0(\mathbf{s}) + i\lambda_0\delta(\mathbf{s})\sin\theta_0(\mathbf{s}).$$

Thus the equation (4.7) is equivalent to the following equation

$$(\delta(\mathbf{s}) - 1)z + \lambda_0 z_0(1 - \delta(\mathbf{s})) + \frac{\lambda_0}{z_0}(1 + \delta(\mathbf{s})) - (1 + \delta(\mathbf{s}))\frac{1}{z} = 0, \tag{4.8}$$

where $z_0 = e^{i\theta_0}$. By simple calculation we obtain that

$$\frac{1}{z} - \frac{\lambda_0}{z_0} = \frac{\delta(\mathbf{s}) - 1}{\delta(\mathbf{s}) + 1}(z - \lambda_0 z_0).$$

Furthermore since z is a pole in $B_2(0,1)$ and $|\dfrac{\delta - 1}{\delta + 1}| < 1$, we obtain

$$|\frac{1}{z} - \frac{\lambda_0}{z_0}| < |z - \lambda_0 z_0|.$$

So, we get

$$|1 - \lambda_0 \varsigma| < |\varsigma - \lambda_0|,$$

where $\varsigma = \dfrac{z}{z_0}$, which means

$$1 - |\varsigma|^2 < \lambda_0(1 - |\varsigma|^2).$$

Hence, we conclude that $\lambda_0 > 1$ which is a contradiction. Thus F has one simple pole $z = 0$ and by direct application of the residue Theorem, we conclude the proof of Proposition 4.2. $\square$

Finally we conclude that:

$$D(\mathbf{s}, U) = \frac{1}{2i\pi} \int_{\mathcal{C}(0,1)} F(z)dz = -\frac{\mathbf{b}(\mathbf{s})}{\mathbf{a}(\mathbf{s}) + \mathbf{b}(\mathbf{s})} w_2(\mathbf{s}) - \frac{\mathbf{a}(\mathbf{s})}{\mathbf{a}(\mathbf{s}) + \mathbf{b}(\mathbf{s})} w_3(\mathbf{s}).$$

This ends the proof of Proposition 4.1.

[7] [2]

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